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SEMIFREDHOLM LINEAR RELATIONS IN OPERATOR RANGLES*

The stability of semiFredholm operators under strictly singular or small perturbation in Banach spaces were studied by several authors [9, 14, 16, 18],. . . . The main purpose of this paper is to study this problem for semiFredholm linear relations in operator ranges. We also give properties of adjoints of semiFredholm linear relations. Examples are exhibited proving that these results are not valid in arbitrary normed spaces.

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§ 1. Introduction

Properties of conjugates and the stability of semiFredholm operators in Banach spaces under strictly singular or small perturbations were studied by several authors [9, 14, 16, 18, . . .]. The aim of this paper is to study these questions in the more general setting of linear relations between operator ranges.

In Section 2, we introduce the class of semiFredholm linear relations in normed spaces and we prove several results which will be used to prove the main results.

In Section 3 we investigate the relationship between a semiFredholm linear relation and its conjugate.

In Section 4 we study the perturbation of semiFredholm linear relations following the pattern followed for study the analogue question of semiFredholm operators.

Notations. We adhered to the notation and terminology of the book [8] : X, Y are normed spaces, X' — the dual space of X.

If $M \subset X$ and $N \subset X'$ are subspaces, then $M^{\perp} := \{x' \in X' : x'(x) = 0 \ for \ all \ \ x \in M\}$, $N^{\top} := \{x \in X : x'(x) = 0 \ for \ all \ x' \in N\}$. A multivalued linear operator or linear relation $T : X \to Y$, denoted by $T \in LR(X, Y)$ or $T \in LR(X)$ when $X = Y$, is a set-valued map such that its graph $G(T) := \{(x, y) \in X \times Y : y \in Tx\}$ is a linear subspace of $X \times Y$. Since $T(0)$ is a linear subspace and $y \in Tx \Leftrightarrow Tx = y + T(0)$, it follows that for x_1,x_2 in the domain of T (which is denoted by $D(T)$), and non-zero scalar α we have $Tx_1 + Tx_2 = T(x_1 + Tx_2)$ and $T(\alpha x_1) = \alpha Tx_1$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply operator.

Let M be a subspace of $D(T)$. Then the restriction $T\mid_M$ is defined by $G(T\mid_M) :=$ $:= \{(m, y) : m \in M,\ y \in Tm\}$. For any subspace M of X such that $D(T) \cap M \neq \varnothing$, we write $T\mid_{M\cap D(T)} = T\mid_M$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) =$ $= \{(y, x) \in Y \times X : (x, y) \in G(T)\}$. If T^{-1} is single valued, then T is called injective, that is, T is injective if and only if its null space $N(T) := T^{-1}(0) = \{0\}$, T is said to be

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surjective if its range $R(T) := TD(T) = Y$. The completion of T, denoted by $\widetilde{T}$, is defined by $G(\widetilde{T}) := \widetilde{G(T)} \subset \widetilde{X} \times \widetilde{Y}$, where $\widetilde{X}$ denotes the completion of X. For $T \in LR(X,Y)$ we denote $a(T) := dimN(T)$, $b(T) := dimY/R(T)$, $\overline{b}(T) := dimY/\overline{R(T)}$. The index of T is defined as $i(T) := a(T) - b(T)$ provided $a(T)$ and $b(T)$ are not both infinite.

The adjoint or conjugate T' of T is defined by $G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'$, where $\langle (y,x),(y',x')\rangle := \langle x,x'\rangle + \langle y,y'\rangle$. This means that $(y',x') \in G(T')$ if and only if $y\prime(y) - x\prime(x) = = 0$ for all $(x,y) \in G(T)$.

For $T \in LR(X,Y)$, Q_T denotes the quotient map from Y onto $Y/\overline{T(0)}$. Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\| Tx \| := \| Q_T\ Tx \|$ and the norm of T is defined by $\| T \| := \| Q_T T \|$.

A linear relation $T \in LR(X,Y)$ is called closed if its graph is a closed subspace, continuous if for each neighbourhood V in $R(T)$, $T^{-1}(V)$ is a neighbourhood in $D(T)$ equivalently $\| T \| < \infty$, open if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \| Tx \| \ for\ x \in D(T)\}$, compact if $\overline{Q_T T B_{D(T)}}$ is compact, precompact if $Q_T T B_{D(T)}$ is totally bounded (here $B_{D(T)}$ denotes the closed unit ball of $D(T)$), strictly singular if there is no infinite dimensional subspace M of $D(T)$ for which $T\mid_M$ is injective and open. Continuous everywhere defined operators referred to as bounded operators.

We call a subset R of a Banach space Y an operator range if there exists a Banach space X and a bounded operator $T:\ X \to Y$ such that $R(T) = R$. We note that a subspace R of a Banach space is an operator range if and only if there exists a stronger norm on R under which R is complete [6]. If $\| \ . \ \|$ is the norm on R we will denote the stronger norm on R under which R is Banach by $\| \ . \ \|_1$. We will denote $(R, \| \ . \ \|_1)$ by R_1. The bounded bijection from R_1 onto R will be denoted by α_R and we will use β_R to denote the closed and open bijection α_R^{-1}.

Linear relations were introduced in Functional Analysis by J. von Neumann [19], motivated by the need to consider of non-densely defined linear differential operators which are considered by Coddington [4], Coddington and Dijksma [5], among others. Problems in optimisation and control also led to the study of set-valued maps and differential inclusions (see, for example, Aubin and Cellina [2], Clarke [3], among others). Studies of convex processes, tangent cones... form part of the theory of convex analysis developed to deal with non-smooth problems in viability and control theory, for example. Others recent works on multivalued mappings include the treatise on partial differential relations by Gromov [15], the application of multivalued methods to solving differential equations by Favini and Yagi [11] and the book "Multivalued Linear Operators" by Cross [8].

Operator ranges are natural objects of investigation because the domain of a closed operator between Banach spaces is an operator range and every operator range is the domain of some closed operator. Many normed spaces that appear in applications are operator ranges, like the spaces C[0,1] with the norm of L_2[0,1], or some Sobolev spaces with suitable L_2-norms. One more reason for their investigation is that the Burnside theorem on invariant subspaces of algebras of operators in finite dimensional spaces admits an adequate generalisation to strongly closed algebras of operators on Hilbert spaces in terms of invariant operator ranges [13]. Operator ranges in Banach spaces were studied in the papers [1, 6, 7, 10, 12, 13, 17, 21, 22], among others. Nevertheless many natural questions remain unanswered.

§ 2. Definitions and basic results

The notion of semiFredholm operator in Banach spaces can be generalised naturally to linear relations in normed spaces as follows:

Definition 1. *Let $T \in LR(X,Y)$ be closed. We say that T is upper semiFredholm if T has closed range and finite dimensional null space, lower semiFredholm if its range is closed and finite codimensional, semiFredholm if T is upper or lower semiFredholm and T is called Fredholm if it is upper and lower semiFredholm.*

The corresponding classes of linear relations will be denoted by $\mathcal{USF}(X,Y)$, $\mathcal{LSF}(X,Y)$, $\mathcal{SF}(X,Y)$ and $\mathcal{F}(X,Y)$ respectively.

In this Section we prove some auxiliary results that we shall need to obtain the main theorems.

Lemma 2. *Let $T \in LR(X,Y)$ be closed. Then*

(i) *$Q_T T$ is a closed operator and $T(0)$ and $N(T)$ are closed subspaces.*

(ii) *$R(T)$ is closed if and only if so is $R(Q_T T)$.*

(iii) *$N(T) = N(Q_T T)$ and $b(T) = b(Q_T T)$.*

(iv) *$R(T') = R((Q_T T)'), a(T') = \overline{b}(T)$ and $a(T) \leq \overline{b}(T')$ with equality if T is open.*

Proof. (i) If T is closed, then by [8, II.5.1 and II.5.3], $Q_T T$ is closed, $T(0)$ is closed and T^{-1} is closed and hence $N(T) := T^{-1}(0)$ is closed.

(ii) Note that $R(Q_T T) = R(T)/T(0)$ is closed if and only if $R(T)$ is closed (as $T(0)$ is closed by (i)).

(iii) That $N(T) = N(Q_T T)$ follows from [8, II.3.4] and (i), while the equality $b(T) = b(Q_T T)$ is a simple consequence of the property $T(0)$ closed and [8, I. 6.10].

(iv) The equality $R(T') = R((Q_T T)')$ holds trivially by [8, III. 1.10]. Finally, as $N(T') = R(T)^{\perp}$ and $N(T) = R(T')^{\top}$ by [8, III. 1.4], we obtain that $a(T') = \dim N(T') = \dim R(T)^{\perp} = \dim(Y/\overline{R(T)})' = \dim Y/\overline{R(T)} = \overline{b}(T)$ and $a(T) = \dim N(T) = \dim N(T)' = \dim X'/N(T)^{\perp} \leq \dim X'/\overline{R(T)} = \overline{b}(T')$. (Note that here $N(T)$ is closed since T is closed). But if T is open, then $N(T)^{\perp} = R(T')$ by [8, III. 4.6] and thus $a(T) = b(T')$. □

Proposition 3. *Let $T \in LR(X,Y)$ be closed. Then we have*

(i) *If X is an operator range, then $T\alpha_X$ is closed, $R(T) = R(T\alpha_X)$ and $a(T) = a(T\alpha_X)$.*

(ii) *If Y is an operator range, then $\beta_Y T$ is closed, $N(T) = N(\beta_Y T)$ and $b(T) = b(\beta_Y T)$.*

Proof. (i) Clearly $R(T\alpha_X) = T\alpha_X D(T\alpha_X) = T\alpha_X \alpha_X^{-1} D(T) = TD(T) = R(T)$.

Assume in the first instance that T is single valued. Then, $T\alpha_X$ is obviously closed since T is closed and α_X is a bounded operator. Clearly, $a(T) = a(T\alpha_X)$. For the general case, substituting T for $Q_T T$, the result follows from Lemma 2 combined with the case when T is assumed to be single valued.

(ii) We consider the canonical factorisation of T, $T = \widehat{T} Q_{N(T)}$, where the injective component $\widehat{T}$ of T is the linear relation $\widehat{T} \in LR(X/N(T), Y)$ given by $G(\widehat{T}) := \{([x], y) :$

$(x,y) \in G(T)\}$ and $Q_{N(T)}$ is the quotient map from X onto $X/N(T)$. In [8, V. 13.2] it is shown that $\widehat{T}$ is closed if and only if so is T. (Note that here $N(T)$ is closed by virtue of Lemma 2). Then $\widehat{T}$ and thus $\widehat{T}^{-1}$is closed and the statement (i) assures that $\widehat{T}^{-1}\alpha_Y = \widehat{T}^{-1}\beta_Y^{-1} = \widehat{(\beta_Y T)}^{-1}$ is closed. Hence $\beta_Y\widehat{T} = \widehat{\beta_Y T}$ and thus $\beta_Y T$ is closed.

Clearly $N(\beta_Y T) := (\beta_Y T)^{-1}(0) = T^{-1}\beta_Y^{-1}(0) = T^{-1}(0) := N(T)$. The equality $b(T) = b(\beta_Y T)$ is just a special case of the following property due to Cross [8, I. 6.11] : Let $U \in LR(X,Y), V \in LR(Y,Z)$ such that $D(V) = Y$. Then $a(VU)+b(U)+b(V)+\dim\{U(0)\cap \cap N(V)\} = b(VU)+a(U)+a(V)$. □

An obvious consequence of Proposition 3 is the following

Corollary 4. *Let $T \in LR(X,Y)$. Then*

(i) *If X is an operator range, then $T \in \mathcal{USF}(X,Y)$ (respectively, $T \in \mathcal{LSF}(X,Y)$) implies that $T\alpha_X \in \mathcal{USF}(X_1,Y)$ (respectively, $T\alpha_X \in \mathcal{LSF}(X_1,Y)$) and $T \in \mathcal{F}(X,Y)$ implies that $T\alpha_X \in \mathcal{F}(X_1,Y)$ with $i(T) = i(T\alpha_X)$ in both cases.*

(ii) *If Y is an operator range, then $T \in \mathcal{USF}(X,Y)$ (respectively, $T \in \mathcal{LSF}(X,Y)$) implies that $\beta_Y T \in \mathcal{USF}(X,Y_1)$ (respectively, $\beta_Y T \in \mathcal{LSF}(X,Y_1)$) and $T \in \mathcal{F}(X,Y)$ implies that $\beta_Y T \in \mathcal{F}(X,Y_1)$ with $i(T) = i(\beta_Y T)$ in both cases.*

We have the following generalisation of the Closed Graph theorem for linear relations [8, III. 5.4].

Proposition 5. *Let $T \in LR(X,Y)$ be closed with $D(T)$ closed in the Banach space X and Y is an operator range. Then T is continuous.*

Proof. By Proposition 3, $\beta_Y T \in LR(X,Y_1)$ is closed with $D(\beta_Y T) = D(T)$ and Y_1 is obviously complete. Then, by [8, III. 5.4], $\beta_Y T$ is continuous. Since $\beta_Y T(0) \subset D(\alpha_Y) = Y_1$, $T = \alpha_Y(\beta_Y T)$ is continuous by [8, II. 3.13]. □

§ 3. Adjoint of semiFredholm linear relations

We now analyse the connection between a semiFredholm linear relation and its conjugate.

We have the following classic result:

Proposition 6. *Let $T \in LR(X,Y)$ be a closed single valued where X and Y are Banach spaces. Then*

(i) *$T \in \mathcal{USF}(X,Y)$ if and only if $T' \in \mathcal{LSF}(Y',X')$.*

(ii) *$T \in \mathcal{LSF}(X,Y)$ if and only if $T' \in \mathcal{USF}(Y',X')$.*

The corresponding properties for linear relations will now be investigated.

Theorem 7. *Let $T \in LR(X,Y)$ with X an operator range and Y complete. Then*

(i) *$T' \in \mathcal{LSF}(Y',X')$ if $T \in \mathcal{USF}(X,Y)$.*

(ii) *$T' \in \mathcal{USF}(Y',X')$ if $T \in \mathcal{LSF}(X,Y)$.*

Proof. We first prove that

"If X is an operator range, Y is complete and $T \in LR(X,Y)$ is closed with $R(T)$ closed, then $R(T')$ is closed". (1)

Indeed, T^{-1} is closed with $D(T^{-1}) = R(T)$ closed in Y Banach. So, from Proposition 5 it follows that T^{-1} is continuous, that is, T is open equivalently $R(T') = N(T)^{\perp}$ [8, III. 4.6]. Hence, the property (1) is established and also that $dimN(T) = dimX'/N(T)^{\perp} =$ $= \dim X'/R(T')$. These facts combined with $\dim N(T') = \dim R(T)^{\perp}$ ([8, III. 1.4]) $=$ $= dimY/R(T)$ yields the desired result. □

The following example illustrates that the completeness of Y is essential in Theorem 7.

Example 8. *Let X be a Banach space. Then there exists a normed space Y and a Fredholm everywhere defined operator $T \in LR(X, Y)$ such that T' is not semiFredholm.*

Let $S \in LR(X)$ be an injective everywhere defined precompact operator, $Y := R(S)$ and let T be the operator S considered as an element of $LR(X, Y)$. Then, it is clear that T is closed, injective and surjective and hence T is Fredholm. However, T' is not semiFredholm since T' is compact.

Theorem 9. *Let $T \in LR(X, Y)$ be closed with X a Banch space. Then*

(i) *$T \in \mathcal{USF}(X, Y)$ if $T' \in \mathcal{LSF}(Y', X')$.*

(ii) *$T \in \mathcal{LSF}(X, Y)$ if $T' \in \mathcal{USF}(Y', X')$.*

Proof. We first verify that

"If X is a Banach space and $T \in LR(X, Y)$ is closed such that $R(T')$ is closed, then $R(T)$ is closed"

To prove this property it is enough to note that $R(T')$ is closed if and only if T' is open ([8, III. 4.2, III. 4.4 and III. 5.2]) and that if X is complete and $T \in LR(X, Y)$ is closed, then $\gamma(T) = \gamma(T')$ and $R(T)$ is closed if $\gamma(T) > 0$ ([8, III. 5.3]). Now, proceeding as in Theorem 7 we obtain that $a(T) = b(T')$ and $a(T') = b(T)$. □

Example 10 below shows that Theorem 9 fails if X is not complete.

Example 10. *There exists a normed space X and a closed operator $T \in LR(X)$ such that T is not semiFredholm and T' is Fredholm.*

Let $X = c_{oo}$ be the space of all scalar sequences which at most finitely many non-zero coordinates normed by the norm $\| (\alpha_n) \| := \sup\{| \alpha_n | : n \in N\}$ and we define S by $S : x := (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in X \to Sx := (0, \alpha_1, \alpha_2/2, \ldots, \alpha_n/n, \ldots) \in X$

Labuschagne [18, 20] proves that S is a precompact operator not compact such that $R(I - S)$ is a proper dense subspace of X. Hence, $T := I - S$ is not semiFredholm and T' is Fredholm (as S' is compact).

§ 4. Perturbation theorems for semiFredholm linear relations

In this Section we analyse the stability of semiFredholm linear relations under strictly singular or small perturbations.

Theorem 11. *Let $T \in LR(X, Y)$ with X an operator range and Y complete. Let $K \in$ $\in LR(X, Y)$ be continuous and strictly singular such that $K(0) \subset T(0)$ and $\overline{D(T)} \subset D(K)$. Then*

(i) *If $T \in \mathcal{USF}(X, Y)$ then $T + K \in \mathcal{USF}(X, Y)$ and $i(T) = i(T + K)$.*

(ii) *If* $T \in \mathcal{LSF}(X,Y)$ *and* $(K\alpha_X)'$ *is strictly singular, then* $T+K \in \mathcal{LSF}(X,Y)$ *and* $i(T) = i(T+K)$.

Proof. We shall first show the following property:

"Let $T \in LR(X,Y)$ be closed and let Y be complete. If $S \in LR(X,Y)$ is continuous with $S(0) \subset T(0)$ and $\overline{D(T)} \subset D(S)$, then $T+S$ is closed". (2)

Suppose first that T and S are single valued and let (x_n) be a sequence in $D(T{+}S) = D(T)$ such that $x_n \to x$ and $(T+S)x_n \to y$ for some $x \in X$ and $y \in Y$. Then $\|T(x_n - x_m)\| \leq \leq \|(T+S)(x_n - x_m)\| + \|S\|\|x_n - x_m\|$. Thus (Tx_n) is a Cauchy sequence in Y and, since Y is complete $Tx_n \to z$ for some $z \in Y$. Since T is closed and S is continuous we obtain that $x \in D(T)$ and $(T+S)x = y$, that is, $T+S$ is closed.

Turning to the general case, it follows from by hypothesis and Lemma 2 that $(T{+}S)(0) = = T(0)$ (so $Q_{T+S} = Q_T$), $Q_T T$ is a closed single valued and $Q_T S$ is a continuous single valued. Applying the first part of the proof, we have that $Q_{T+S}(T+S) = Q_T T + Q_T S$ is closed and again by Lemma 2 it follows that $T+S$ is closed, as desired.

(i) Let $T \in \mathcal{USF}(X,Y)$. Assume in the first instance that T and K are single valued. The assertion follows immediately combining Proposition 3 with the strictly singular perturbation theorem for upper semiFredholm operators in Banach spaces (see, for example [14, V. 2.1]), applied to $T\alpha_X$ and $K\alpha_X$.

For the general case, we observe that $(T+K)(0) = T(0)$, $Q_T T$ is a closed operator and $Q_T K$ is a continuous strictly singular operator. Thus, what has already been proved, $Q_{T+K}(T+K) = Q_T T + Q_T K$ verifies the desired property and by virtue of Lemma 2 the same is true to $T+K$.

(ii) Suppose (ii) holds. By the Open Mapping theorem we have that $T\alpha_X$ is open if and only if $R(T\alpha_X)$ is closed if and only if $R((T\alpha_X)')$ is closed. From this fact combined with Lemma 2 and Proposition 3 we see that $R(T) = R(T\alpha_X)$, $a(T) = a(T\alpha_X) = b((T\alpha_X)')$ and $b(T) = b(T\alpha_X) = a((T\alpha_X)')$. Therefore $(T\alpha_X)' \in \mathcal{USF}(Y',X')$.

Since $K\alpha_X$ is continuous and strictly singular, we obtain that $\overline{D((T\alpha_X)')} \subset D((K\alpha_X)')$ and $(K\alpha_X)'(0) = D(K\alpha_X)^{\perp}$ ([8, III. 1.4])$\subset \overline{(D(T\alpha_X))}^{\perp} = (T\alpha_X)'(0)$. Thus, applying (i) to $(T\alpha_X)'$ and $(K\alpha_X)'$ we deduce that $(T\alpha_X)' + (K\alpha_X)' \in \mathcal{USF}(Y',X_1')$ with $i((T\alpha_X)') = = i((T\alpha_X)' + (K\alpha_X)')$. But, since $D(T\alpha_X) \subset D(K\alpha_X)$ and $K\alpha_X$ is continuous, $(T\alpha_X)' + + (K\alpha_X)' = ((T+K)\alpha_X)'$ by [8, III. 1.5].

Therefore $((T+K)\alpha_X)'$ is upper semiFredholm equivalently $(T+K)\alpha_X$ is lower semiFredholm by Proposition 6 with $i(T\alpha_X) = i((T+K)\alpha_X)$. The result now follows immediately by considering Proposition 3. □

The above Example 10 shows that the class of semiFredholm linear relations in arbitrary normed spaces is not stable under strictly singular perturbations.

Finally we study the behaviour of semiFredholm linear relations under small perturbation. Recall that a linear relation $T \in LR(X,Y)$ is called a F_+-relation if there exists a finite codimensional subspace M of X for which $T\mid_M$ is injective and open and T is said to be a F_--relation if its adjoint is a F_+-relation, [8, V. 1.1].

Proposition 12. *Let* $T \in F_+(X,Y) \cup F_-(X,Y)$ *and suppose that* $K \in LR(X,Y)$ *satisfies* $\overline{D(T)} \subset D(K)$, $K(0) \subset \overline{T(0)}$ *and* $\| K \| < \gamma(\widetilde{T})$. *Then* $T+K \in F_+(X,Y) \cup F_-(X,Y)$ *and*

$i(\widetilde{T}) = i(\widetilde{T} + \widetilde{K})$.

Proof. This result was proved by Wilcox [23, 6. 1.1]. The proof is along the lines of the proof of the analogous result provided in [8, V. 15.6] for the case when K is single valued. □

Theorem 13. *Let X be an operator range, Y complete, $T \in LR(X,Y)$, $K \in LR(X,Y)$ such that $K(0) \subset \overline{T(0)}$, $\overline{D(T)} \subset D(K)$ and $\| K \| < 1/ \| \alpha_X \| \gamma(T\alpha_X)$. Then*

(i) *If $T \in \mathcal{USF}(X,Y)$, then $T + K \in \mathcal{USF}(X,Y)$ and $i(T) = i(T+K)$.*

(ii) *If $T \in \mathcal{LSF}(X,Y)$, then $T + K \in \mathcal{LSF}(X,Y)$ and $i(T) = i(T+K)$.*

Proof. By virtue of [8, V. 1.7] and Theorem 7 we deduce that for closed linear relations in Banach spaces we have the equivalences:

$$T \in F_+ \Leftrightarrow T \in \mathcal{USF} \text{ and } T \in F_- \Leftrightarrow T \in \mathcal{LSF} \quad (3)$$

(i) Suppose that T is upper semiFredholm. Then, from (3) and Proposition 3, $T\alpha_X$ is a F_+-relation and thus applying Proposition 12 to $T\alpha_X$ and $K\alpha_X$ we obtain that $(T+K)\alpha_X$ is upper semiFredholm and $i(T\alpha_X) = i((T+K)\alpha_X)$. The assertion now follows immediately from Proposition 3.

(ii) If T is lower semiFredholm, then proceeding as in the proof of Theorem 11 and observing that $\| K\alpha_X \| = \| (K\alpha_X)' \| < \gamma(T\alpha_X) = \gamma((T\alpha_X)')$ we conclude that $T + K$ is lower semiFredholm and $i(T) = i(T+K)$. □

The following example shows that the class of semiFredholm linear relations in arbitrary normed spaces is not stable under small perturbations even for bounded operators.

Example 14. *There exists a normed space X and a continuous everywhere defined operator $T \in LR(X)$ such that $\| T \| < 1$ and for any $0 < | \lambda | < 1$, $R(I + \lambda T)$ is not closed.*

Let $X = c_{oo}$ and let T be the right-shift operator. For any scalar λ such that $0 < | \lambda | < 1$, we have that $\| \lambda T \| < 1 = \gamma(I)$ and Labuschagne [18, 6.1] proves that $R(I + \lambda T)$ is a dense subspace of c_{oo}.

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