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РЕЛЯТИВИСТСКИЕ ЭФФЕКТЫ В РАВНОВЕСИИ ЗАРЯЖЕННОЙ ПЛАЗМЫ

Показано, что релятивистские эффекты существенным образом изменяют равновесную конфигурацию заряженной плазмы, даже если линейная скорость вращения плазмы в скрещенных электрическом и магнитном полях мала по сравнению со скоростью света. Изменение особенно выражено в режиме быстрого вращения, когда частота азимутального вращения приближается к циклотронной частоте. При этом радиальный профиль плотности плазмы становится приблизительно параболическим в отличие от ступенчатого, который предсказывается нерелятивистской теорией. Отмечено, что указанные эффекты могут быть обнаружены в экспериментах, подобных эксперименту Тесса и др. [Phys. Rev. Lett. =135, 1436 (1975)]. Обсуждается также изменение бриллюэновского предела плотности плазмы в релятивистском случае.

Ключевые слова: заряженная плазма, равновесие, релятивистские эффекты.

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Relativistic Equilibrium of Nonneutral Plasmas

It is shown that relativistic effects strongly modify the equilibrium of nonneutral plasmas even if the linear velocity of plasma rotation in crossed electric and magnetic fields is small as compared to the speed of light. The change is especially pronounced for the fast rigid-rotor equilibrium, when the frequency of the azimuthal rotation is close to the cyclotron frequency, and the radial density profile becomes approximately parabolic rather than stepwise as predicted by the non-relativistic theory. It is argued that such effects could be detected in experiments similar to those performed by Theiss et al. [Phys. Rev. Lett. 135, 1436 (1975)]. The relativistic modification of the Brillouin density limit is also addressed.

Introduction

As is well known [1], within a non-relativistic cold fluid model the equilibrium of an infinitely long nonneutral plasma column with constant density, confined radially by a uniform magnetic field B_0 is characterized by the azimuthal rotation frequencies

$$\omega = \omega^{\pm} = -\frac{1}{2}\Omega\left[1 \pm \left(1 - 2\omega_p^2/\Omega^2\right)^{1/2}\right], \qquad (1)$$

where $\omega_p = (4\pi e^2 n/m)^{1/2}$ is the plasma frequency, with n, e and m the particle density, charge and mass, respectively, and $\Omega = eB_0/mc$ is the non-relativistic cyclotron frequency (Ω is assumed to have the same sign of the charge). The two angular frequencies ω^- and ω^+ correspond to a slow and a fast rotation of the plasma column, respectively. As $\omega^{\pm}$ are independent of the radial coordinate r, the azimuthal motion of the plasma column corresponds to a rigid rotation about the axis of symmetry.

For a low density plasma, $2\omega_p^2/\Omega^2 \ll 1$, the frequency of slow rotation is approximately equal to the electric drift (diocotron) frequency,

$\omega^- \simeq -\omega_p^2/2\Omega$, while the frequency of fast rotation approaches the cyclotron frequency, $\omega^+ \simeq -(\Omega - \omega_p^2/2\Omega)$. For $2\omega_p^2/\Omega^2 = 1$, the two rotational equilibria merge, and $\omega^+ = \omega^- = -\Omega/2$. The condition $2\omega_p^2/\Omega^2 = 1$ is usually referred to as the Brillouin density limit: radially confined equilibria do not exist for $n > n_{B0} \equiv B_0^2/(8\pi mc^2)$ [1, 2]. The two rotational equilibria have been measured experimentally [3].

It is usually assumed that the electric current due to the azimuthal rotation of the nonneutral plasma column produces a negligible change of the axial magnetic field. A simple estimate readily shows that for the slow rotation equilibrium the relative depression of the magnetic field can be neglected,

$$\frac{\delta B}{B} \simeq \left(\frac{\omega_p^2}{2\Omega}\right)^2 \frac{r^2}{c^2} \ll 1, \qquad (2)$$

as long as the plasma radius is much smaller than the maximum radius of a rigidly rotating body allowed by the theory of relativity, $r \ll c/|\omega^-|$ [4]. The inequality (2) means that relativistic effects are unessential for the slow rotation mode The diamagnetic correction of the magnetic field in cold nonneutral plasma is smaller than the relativistic correction (2) by the square of the ratio r_p/λ_D of the plasma radius r_p to the Debye length $\lambda_D = (T/4\pi e^2 n)^{1/2}$. It is shown below, however, that this may be not true for the fast rotation equilibrium.

Besides the variation of the magnetic field by plasma currents, another effect that must be taken into account is the relativistic modification of the centrifugal force. In the slow rotating equilibrium, the centrifugal force plays no significant role being much smaller than both Lorentz's and electric forces. On the contrary, in the fast rotating equilibrium it is almost equal to the Lorentz force while the electric force is small, being of the order of the relativistic corrections in the limit $n \ll n_{B0}$.

Cold relativistic equilibrium

For cold plasma, the relativistic macroscopic fluid radial force balance equation is written as

$$-\gamma \frac{mv_\theta^2}{r} = eE_r + \frac{e}{c} v_\theta B \qquad (3)$$

where $\gamma = (1 - v_0^2/c^2)^{-1/2}$ is the relativistic factor, and

$$r\frac{d}{dr}(rE_r) = 4\pi en, \qquad (4)$$

$$\frac{dB}{dr} = -\frac{4\pi}{c} env_\theta. \qquad (5)$$

Integrating the preceding equations with $v_\theta = r\omega$, one obtains [1]

$$\gamma\omega^2 r + \frac{1}{r}\int_0^r \omega_p^2\, x\, dx + \Omega\omega r - \frac{\omega r}{c^2}\int_0^r \omega_p^2 \omega x\, dx = 0. \qquad (6)$$

This equation can be solved with respect to ω for a given radial density distribution, or vice versa a solution for the function $\omega_p^2(r)$ can be sought for for a given radial profile of the rotation frequency, $\omega(r)$. The case of rigid rotation, $\omega(r) = \text{const}$, is of special interest as it characterizes a state of global thermal equilibrium, considered in Refs. 5, 6 for the non-relativistic case. The relativistic cold global equilibrium is analyzed in the next Section.

Rigid rotor equilibrium

When ω is independent of r, Eq. (6) reads

$$2\nu\rho^2 + \frac{2\nu^2\rho^2}{\sqrt{1-\rho^2}} + (1-\rho^2)\int_0^\rho xN(x)\, dx = 0, \qquad (7)$$

where $\rho \equiv r|\omega|/c$, $\nu \equiv \omega/\Omega$, and $N \equiv 2\omega_p^2/\Omega^2 = n/n_{B0}$. Dividing Eq. (7) by $(1-\rho^2)$ and differentiating with respect to ρ gives

$$N(\rho) = -\frac{4\nu}{(1-\rho^2)^2} - \frac{2\nu^2(2+\rho^2)}{(1-\rho^2)^{5/2}}. \qquad (8)$$

Eq. (5) then yields

$$\frac{B}{B_0} = \frac{1}{1-\rho^2} + \frac{\nu\rho^2}{(1-\rho^2)^{3/2}}. \qquad (9)$$

The non-relativistic limit is obtained by simply putting $\rho = 0$ everywhere in Eqs. (8) and (9). Figs. 1 and 2 show radial profiles of the density, Eq. (8), and the axial magnetic field,

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[1] Eq. (6) differs from Eq.(5.19) in Ref. 1 by the definition of Ω through the magnetic field on the axis, rather than through the externally imposed magnetic field. This simplifies formulas (8) and (9) in comparison to their counterparts (5.22) and (5.23) in Ref. 1.

Eq. (9), respectively, for different values of the normalized rotation frequency ν.

Physically acceptable solutions correspond to $-1<\nu<0$, since otherwise $N(\rho)$ becomes negative everywhere. The actual density profile is given by Eq. (8) for $0<\rho<\rho_p<\rho_0$, where ρ_p represents the (normalized) value of the plasma radius r_p, and ρ_0 denotes the limiting radius where $N(\rho)=0$. The latter can be given explicitly as

$$\rho_0=\left[\frac{2(1-\nu^2)}{(1+3\nu^2)^{1/2}+1+\nu^2}\right]^{1/2}. \tag{10}$$

For $\nu\to-1$, the density profile becomes parabolic,

$$N(\rho)\simeq 4(\nu+1)-4\rho^2, \tag{11}$$

and Eq. (10) gives $\rho_0\simeq\sqrt{1+\nu}\to 0$. In dimensional units the limiting radius of the rigid-rotor equilibrium,

$$r_0\simeq\frac{c}{|\Omega|}\left(1+\frac{\omega}{\Omega}\right)^{1/2}\simeq\frac{c\,\omega_{p0}}{\sqrt{2}\Omega^2}, \tag{12}$$

shrinks to zero for $n\ll n_{B0}$, being proportional to the plasma frequency ω_{p0} evaluated with the density at $\rho=0$. Under these conditions the relativistic correction of the centrifugal force is greater by a factor n_{B0}/n than the axial magnetic field variation due to the azimuthal plasma current.

The entire parabolic profile (11) of the fast rotational equilibrium can be observed experimentally, when the radius r_p of the trapped nonneutral plasma approaches the limiting value r_0. This condition can be easily achieved with a suitable choice of the parameters as will be shown in the concluding section.

For $\nu\to 0^-$, $\rho_0\simeq 1-9\nu^2/8$. Hence, in the slow rotational equilibrium the limiting radius is quite large if $n\ll n_{B0}$, being limited only by the maximum radius of a rigidly rotating frame that can be made up by real bodies,

$$r<r_0\simeq\frac{c}{|\omega|}\simeq\frac{2c|\Omega|}{\omega_{p0}^2}. \tag{13}$$

However, the equilibrium density profiles are not stepwise as predicted by classical theory, being peaked near the limiting radius as shown in Fig. 1.

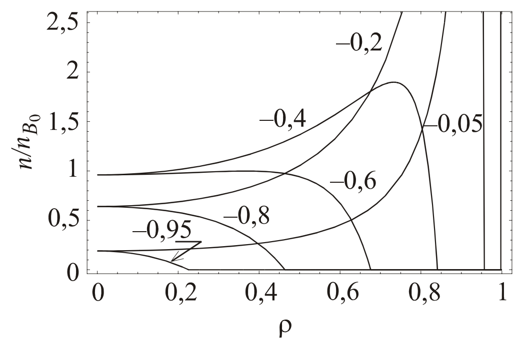

Fig. 1. (Color online) Nonneutral plasma density, normalized over the Brillouin density value n_{B0} evaluated with the magnetic field on the axis, vs. the normalized radius ρ for various values of the normalized rotation frequency ν, indicated on the plot

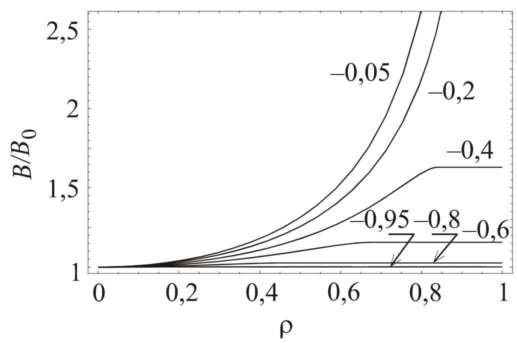

Fig. 2. (Color online) Axial magnetic field vs. ρ for different rotation frequencies ν, indicated on the plot. For $\rho>\rho_0(\nu)$, B corresponds to the externally applied uniform magnetic field

The density profile (8) assumes its maximum value at

$$\rho_{max}=\begin{cases}0\\ \left[\dfrac{4(4-9\nu^2)}{2(16+45\nu^2)^{1/2}+8+9\nu^2}\right]^{1/2}\end{cases}$$

$-1<\nu\le-2/3$,
$-2/3\le\nu<0$, (14)

i.e., the plasma density monotonically decreases with the radius for $-1<\nu<-2/3$, while hollow density profiles are found for $-2/3<\nu<0$. Both monotonic and hollow density profiles shown in Fig. 1 are stable since they correspond to global equilibrium states (at $T=0$), which are not destroyed by like-particle collisions [5].

Brillouin density limit

By definition, the Brillouin limit represents the maximum density, n_{max}, that can be achieved within the allowed range of rigid-rotor frequencies. In the non-relativistic case, the limit is given by $n_{max} = n_{B0}$. However Fig. 1 clearly shows that n_{max}/n_{B0} can be greater than 1. This is especially evident for $\nu \to 0^-$ when the magnetic field shielding becomes very strong, as shown in Fig. 1.

On the other hand, the plasma density remains always smaller than the local Brillouin's density limit, i.e., $n(\rho) \le n_B = n_{B0} \times [B(\rho)/B_0]^2$, as shown in Fig. 3. For a given frequency of rotation the ratio n/n_B is maximal at $\rho = 0$ if $-1 < \nu < -1/2$, and at bigger radii if $-1/2 < \nu < 0$. The absolute maximum of n/n_B is reached for $\nu = -1/2$ and $\rho = 0$, and it is equal to 1.
Thus, the inequality

$$n \le n_B \tag{15}$$

holds locally for every radius and every rotation frequency (see Fig. 4), and can therefore be thought of as a relativistic generalization of the Brillouin density limit for rigidly rotating nonneutral plasmas. Differential rotation generally makes it possible for the plasma density to exceed the Brillouin limit as will be shown in the next section.

Differential rotation

For a non-uniform rotation frequency, it is useful to introduce a different normalization of the lengths; therefore in this section the normalized radius is defined as $\rho \equiv r|\Omega|/c$. Eq. (6) then takes the form

$$\frac{\rho\nu^2}{\sqrt{1-\rho^2\nu^2}} + \rho\nu - \frac{\rho\nu}{2}\int_0^\rho xN(x)\nu(x)\,dx + \\ + \frac{1}{2\rho}\int_0^\rho xN(x)\,dx = 0. \tag{16}$$

For an arbitrary $N(\rho)$ the integral equilibrium equation (16) allows one to determine $\nu(\rho)$. Alternatively, one can seek $N(\rho)$ for a given $\nu(\rho)$. Since Eq. (16) is linear with respect to $N(\rho)$, the latter approach is usually more effective.

Some general properties of the solutions can be deduced by using power series for ν and N. If $N(r=0) \ne 0$, it results from (16) that both ν and N can be expanded in even powers of ρ (with one notable exception, see below). Substituting $\nu(\rho) = \sum_{k=0}^{+\infty} \nu_k \rho^{2k}$ and $N(\rho) = \sum_{k=0}^{+\infty} N_k \rho^{2k}$ into Eq. (16) and gathering the terms with the same power of ρ yields a chain of equations that allow to express N_k recursively through N_{k-1}. At lowest order, this procedure leads to the relation

$$N_0 = -4\nu_0(1+\nu_0), \tag{17}$$

that yields two frequencies, $\nu_0^{\pm}$, for a given density N_0 in agreement with the existence of two equilibria with slow and fast rotation. These two frequencies are the same as the non-relativistic solutions in Eq. (1) (when expressed in dimensional units).

Applying this procedure to the case of uniform density, $N(\rho) = N_0$, one easily obtains an approximate expression

$$\nu^{\pm}(\rho) = \nu_0^{\pm} - \frac{(\nu_0^{\pm})^3(3\nu_0^{\pm}+2)}{2(2\nu_0^{\pm}+1)}\rho^2 + O(\rho^4) \tag{18}$$

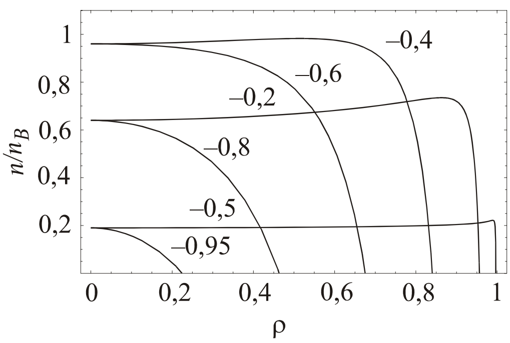

Fig. 3. (Color online) Nonneutral plasma density, normalized over $n_B = B^2/8\pi mc^2$, vs. the normalized radius ρ, for various values of the normalized rotation frequency ν, indicated on the plot

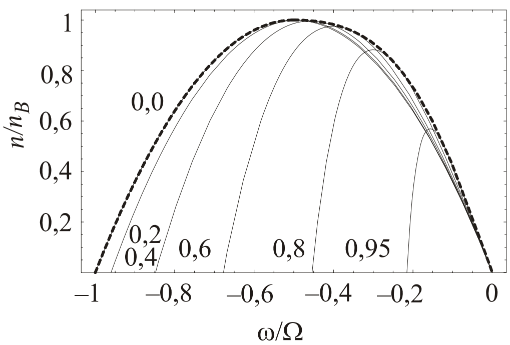

Fig. 4. (Color online) Nonneutral plasma density, normalized over the local Brillouin density limit $n_B = B^2/8\pi mc^2$, for different radii $r|\omega|/c$, indicated on the plot (solid curves), and the maximum of n/n_B (dashed envelope) vs. the rotation frequency

for $\nu(\rho)$ which is valid for small ρ provided that $\nu_0^{\pm}$ is not too close to $-1/2$, i.e. $N_0 \neq 1$. For $N_0 \to 1$, the expansion in even powers of ρ is no longer valid, and a more accurate estimate gives

$$\nu^{\pm}(\rho) = -\frac{1}{2} \mp \left[\left(\frac{1}{2} + \nu_0^{\pm} \right)^2 + \frac{1}{32}\rho^2 \right]^{1/2}. \quad (19)$$

From a practical point of view it is more convenient to deal with a differential equation rather than the original integral equation (16). Namely, a differential equation of the form

$$N' + \tilde{\alpha} N + \tilde{\beta} = 0, \quad (20)$$

can be derived by suitably differentiating Eq. (16) twice, where the prime stands for the derivative with respect to ρ, and $\tilde{\alpha}$ and $\tilde{\beta}$ are functions of ρ, ν, ν' and ν''. Alternatively, one can obtain an equation of the form

$$\nu' + \tilde{\gamma} = 0, \quad (21)$$

with $\tilde{\gamma}$ being a function of ρ, ν, N and $\int_0^{\rho} x N(x) dx$. Both (20) and (21) are first order ordinary differential equations (with respect to the functions $N(\rho)$ and $\nu(\rho)$, respectively), which can be solved, in general numerically, by standard methods. Eqs. (20) and (21) have a wider class of solutions then the original integral equation (16). For example, Eq. (20) generally admits a solution for any boundary condition $N(0) = N_0$, but only the solutions satisfying the relation (17) between $N_0 = N(0)$ and $\nu_0 = \nu(0)$ are acceptable, since they also obey the original integral equation (16). Similar considerations lead to the conclusion that for any given density profile two rotational modes exist accordingly to two solutions $\nu_0^{\pm}$ of Eq. (16).

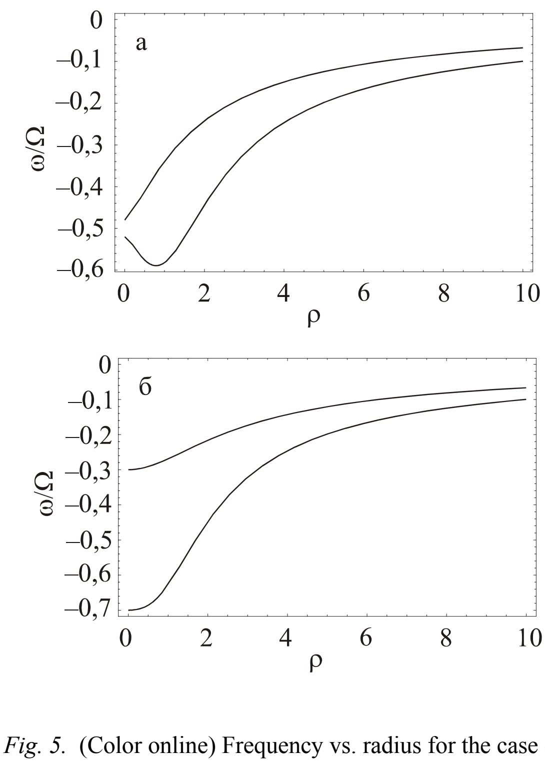

Fig. 5. (Color online) Frequency vs. radius for the case of uniform density: (a)— $\omega_0^{\pm}/\Omega = -(0.5 \pm 0.02)$, the plasma density at the axis is close to Brillouin's limit, $n(0)/n_{B0} = 0{,}998$; (b)— $\omega_0^{\pm}/\Omega = -(0.5 \pm 0.2)$, $n(0)/n_{B0} = 0.84$

The two rotational modes found numerically for the case of uniform density are shown in Fig. 5 for two different values of N_0.

For the fast rotational equilibrium, it can be deduced from Eq. (18) that the frequency profiles are non-monotonic (as in Fig. 5 on the left) for $-2/3 < \nu < -1/2$. Based on the non-relativistic theory, these equilibria are expected to be unstable, since the derivative $d^2\left(\rho^2\nu\right)/d\left(\rho^2\right)^2$ changes sign, while only monotonic profiles are proven to be certainly stable [7,8]. To the best of the author's knowledge, a consistent theory of the stability of nonneutral plasma equilibria with relativistic effects included is not available so far (except for the slow rotational mode—see [9]), so this problem deserves further analysis.

The example of uniform density explicitly demonstrates that the plasma may in principle be unbounded since the limiting radius turns out to be infinite. For any value of N_0, the asymptotic behavior of the rotation frequency for $\rho \to +\infty$ is given by

$$\nu^{+} \simeq -\frac{1}{\rho}, \qquad \nu^{-} \simeq -\frac{1}{\sqrt{2}\rho}.$$

Radially bounded solutions are not an exclusive feature of rigidly rotating plasmas. The following completely analytical solution illustrates this fact, although it is singular at the limiting radius $r_0 = 4c/|\Omega|$:

$$\nu(\rho) = -\frac{1}{2}\frac{1}{1+(\rho/4)^2}, \quad (22)$$

$$N(\rho) = \frac{1+(\rho/4)^2}{[1-(\rho/4)^2]^3}. \quad (23)$$

Despite the singularity, the modified Brillouin limit (15) holds in the entire range $r < r_0$.

Another peculiar solution

$$\nu(\rho) = -\frac{a}{\rho}\frac{1}{\sqrt{1+a^2}}, \tag{24}$$

$$N(\rho) = 2\sqrt{1+a^2}\left(\rho/a\right)^{a^2-1} H(\rho - a) \tag{25}$$

gives an example of annular vortex with internal radius $a > 0$ (H denotes Heaviside's step function). This fast rotational equilibrium is characterized by $B/B_0 = (\rho/a)^{a^2}$ and constant relativistic factor $\gamma = \sqrt{1+a^2}$. For the particular case $a = 1$ the solution describes an annular vortex with uniform density

$$\nu(\rho) = -1/\sqrt{2}\rho, \qquad N(\rho) = 2\sqrt{2}H(\rho - 1). \tag{26}$$

This fast rotational equilibrium shows also that in the case of differential rotation the plasma density can exceed the Brillouin limit found for the case of rigid rotation, see inequality (15).

Discussion

It has been shown that the relativistic change of the centrifugal force and the shielding of the magnetic field by the azimuthal current due to the rotating charged column strongly modify the equilibrium density profile of a nonneutral plasma even if the linear velocity of the rotation is small as compared to the speed of light.

The modification becomes especially strong for the fast rigid rotor plasma equilibrium, when the azimuthal rotation frequency approaches the cyclotron frequency. In this case, the radial extent of the plasma column, allowed by the relativistic effects, turns out to be much smaller than the limiting radius, $c/|\omega|$, of a rigidly rotating frame that can be realized by real bodies. In addition, for the experimentally relevant case in which the limiting radius becomes comparable with the plasma radius r_p, the density profile becomes nearly parabolic rather than stepwise as predicted by the non-relativistic theory, see Eq. (11).

The relativistic modification of the equilibrium density profile should be observable in experiments similar to those performed by Theiss et al. [3] more than 30 years ago. Putting $n \simeq 5\cdot 10^7$ cm^{-3} and $B \simeq 150$ gauss ($2\omega_p^2/\Omega^2 \simeq 0.035$) into Eq. (12) gives $r_0 \le 1.0\,\text{cm}$. For these parameters, thermal effects are negligible for $T \le 1\text{eV}$, since $r_0/\lambda_D \approx 10$. In the earlier experiments [3], the plasma radius was ten times smaller than the limiting radius, so the authors did not notice the modification of the density profile.

The radial extent of the nonneutral plasma column in the case of differential rotation may not be bounded in principle, as it has been demonstrated explicitly for the case of uniform plasma density. However, a non-uniformly rotating plasma column may not be stable in contrast to the case of rigid rotation. Moreover, like-particle collisions tend to eliminate radial gradients of the rotation frequency. It is therefore expected that transport processes or various instabilities lead to the formation of a rigidly rotating, radially bounded equilibrium.

Finally, it has been found that the Brillouin density limit should be modified if the shielding of the external magnetic field by the current associated with the plasma rotation is significant. The modified Brillouin limit has been casted into the form (15) that relates the density of rigidly rotating plasma at a given radius with the magnetic field value at the same radius.

In conclusion, it is worth noting that a correct self-consistent treatment of the plasma density profile close to the plasma edge requires taking into account finite temperature corrections in the force balance equation, and relativistic effects are expected to play a significant role modifying the solution that has been found earlier for the non-relativistic case [5]. The investigation of the relativistic warm plasma equilibrium is currently under way and will be reported elsewhere.

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